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# Free vibrations of plate using two variable refined plate theory

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**Abstract**

The novelty of this paper is the use of two variable refined plate theory for free vibration analysis of plate. Unlike any other theory, the theory is variationally consistent and gives two governing equations, which are only inertially coupled and there is no elastic coupling at all. Number of unknown functions involved is only two, as against three in case of Mindlin's theory. The theory has strong similarity with classical plate theory in many aspects. The theory does not require shear correction factor and transverse shear stress variation is parabolic across the thickness. Simple variant of the theory is also presented.


## 1. Introduction

It is now well known that for vibration analysis of plates, shear deformation effects become important particularly for thick plates or even for thin plates vibrating at higher modes. As classical plate theory (CPT) does not take into account shear effects, many theories have evolved to address the deficiency.

It is worthwhile to note some developments in the plate theory.

Reissner [1,2] was the first to develop a theory which incorporates the effect of shear. Reissner used stress based approach. Later, while at the same level of approximation, Mindlin [3] employed displacement based approach. As per Mindlin's theory, transverse shear stress is assumed to be constant through the thickness of the plate, but this violates the shear stress free surface conditions. Mindlin's theory satisfies constitutive relations for transverse shear stresses and shear strains in an approximate manner by way of using shear correction factor. A good discussion about Reissner's and Mindlin's theories is available in a paper by Wang et al. [4].

Librescu's [5] approach makes the use of weighted lateral displacement. Constitutive relations between shear stress and shear strain are satisfied. Reissner's formulation comes out as a special case of Librescu's [5] approach.

Donnell's [6] approach is to make correction to the classical plate deflections. Donnell assumes uniform distribution of shear force across the thickness of the plate, and, to rectify the effects of the assumption, introduces a numerical factor, which needs to be adjusted.

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**Nomenclature**

| | |
|---|---|
| $a$ | length of a plate in $x$-direction |
| $b$ | width of a plate in $y$-direction |
| $D$ | plate rigidity |
| $E$ | modulus of elasticity of plate material |
| $G$ | shear modulus of plate material |
| $h$ | thickness of a plate |
| $m, n$ | integers, can have values from 1, 2 to $\ldots\infty$ |
| $M_x, M_y, M_{xy}$ | moments due to stresses $\sigma_x$, $\sigma_y$ and $\tau_{xy}$, respectively |
| $Q_x, Q_y$ | shear forces due to stresses $\tau_{zx}$ and $\tau_{yz}$, respectively |
| $t$ | time variable |
| $t_1, t_2$ | values of time variable at the start and end of time interval (in the context of Hamilton's principle), respectively |
| $T$ | kinetic energy |
| $u, v, w$ | displacements in $x$, $y$, and $z$ directions, respectively |
| $u_b, v_b, w_b$ | bending components of displacements $u$, $v$ and $w$, respectively |
| $u_s, v_s, w_s$ | shear components of displacements $u$, $v$ and $w$, respectively |
| $U$ | strain energy |
| $W_{b_{mn}}, W_{s_{mn}}$ | constants associated with mode shapes |
| $x, y, z$ | Cartesian coordinates |
| $0$–$x$–$y$–$z$ | right-handed Cartesian coordinate system |
| $\alpha_{mn}$ | non-dimensional term associated with each $m$ and $n$ |
| $\gamma_{xy}$, $\gamma_{yz}$, $\gamma_{zx}$ | shear strains |
| $\delta$ | symbol representing variational operator |
| $\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$ | normal strains |
| $\mu$ | Poisson's ratio of plate material |
| $\rho$ | mass per unit volume of plate material |
| $\sigma_x$, $\sigma_y$, $\sigma_z$ | normal stresses |
| $\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$ | shear stresses |
| $\omega_{mn}$ | circular frequency of vibration of plate |
| $\overline{\omega}_{mn}$ | non-dimensional frequency of plate, $\omega_{mn}h\sqrt{\rho/G}$ |
| $\nabla^2$ | Laplace operator in two (i.e. $x$ and $y$) dimensions |

Levinson's formulation [7] is based on displacement approach and his theory does not requires shear correction factor. The governing equations for the motion of a plate obtained by Levinson's approach are same as those obtained by Mindlin's theory, provided that the shear coefficient value associated with the Mindlin's theory is taken as $\frac{5}{6}$.

Some important higher-order theories are available in the literature e.g., theories by Murty [8] with 5, 7, 9, … unknowns; Lo et al. [9] with 11 unknowns; Kant [10] with six unknowns; Bhimaraddi and Stevens [11] with five unknowns; Reddy [12] with five unknowns; Soldatos [13] with three unknowns; Reddy [14] with eight unknowns; Hanna and Leissa [15] with four unknowns.

It is important to note that Srinivas et al. [16] have carried out a three-dimensional linear, small deformation theory of elasticity solution for the free vibration of simply-supported, homogeneous, isotropic, thick rectangular plates.

A critical review of plate theories is given by Vasil'ev [17]. Whereas Liew et al. [18] surveyed plate theories particularly applied to thick plate vibration problem. A recent review paper is by Ghugal and Shimpi [19].

It is to be noted that Shimpi [20] presented a theory for isotropic plates. In Ref. [20], the theory was applied to flexure of shear-deformable isotropic plates. The theory using only two unknown functions gave rise to two governing equations, which were uncoupled. Also, unlike many other theories, the theory has strong similarities with the CPT in some aspects. In this paper, the theory has been extended for free vibrations of isotropic plate.

The proposed refined plate theory (RPT) utilizes two components for representing transverse displacement, viz. bending component and shear component. The concept of bending component and shearing component exists in literature for beams in Refs. [21–24].

Anderson [21] and Miklowitz [22] deal with the first-order shear deformable beam theory. They proposed a method based on breakdown of the total deflection into its bending and shear components, for deriving the dynamic solutions of Timoshenko beam theory, to get more convenient form of governing equations.

Further Plantema [25] applied similar approach for analysis of sandwich plates, however, the method used was variationally inconsistent.

It is to be noted that, the aforementioned theories for beams [21,22] and for plates [25], assume that transverse shear stress is constant through the thickness of the plate.

Senthilnathan et al. [26] proposed a higher-order shear deformable theory using similar approach of representing transverse displacement using two components.

The proposed theory is variationally consistent and gives two governing equations, which are only inertially coupled and there is no elastic coupling at all. The theory has strong similarity with CPT in many aspects. The theory and its simple variant is discussed further.

## 2. Plate under consideration

Consider a plate (of length $a$, width $b$, and thickness $h$) of a homogeneous isotropic material. The plate occupies (in 0–$x$–$y$–$z$ right-handed Cartesian coordinate system) a region

$$0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant b, \quad -h/2 \leqslant z \leqslant h/2. \tag{1}$$

The plate can have any meaningful boundary conditions at edges $x = 0$, $a$ and $y = 0$, $b$. The density of the plate material is $\rho$. The modulus of elasticity $E$, shear modulus $G$, and Poisson's ratio $\mu$ of the plate material are related by $G = E/[2(1+\mu)]$.

## 3. RPT for plate vibration

### 3.1. Assumptions of RPT

Assumptions of RPT would be as follows:

1. The displacements ($u$ in $x$-direction, $v$ in $y$-direction, $w$ in $z$-direction) are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.
   As a result, normal strains $\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$ and shear strains $\gamma_{xy}$, $\gamma_{yz}$, $\gamma_{zx}$ can be expressed in terms of displacements $u$, $v$, $w$ by using strain–displacement relations:

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned}\right\}. \tag{2}$$

2. The lateral displacement $w$ has two components: bending component $w_b$ and shear component $w_s$. Both the components are functions of coordinates $x$, $y$ and time $t$ only:

$$w(x, y, t) = w_b(x, y, t) + w_s(x, y, t). \tag{3}$$

3. (a) In general, transverse normal stress $\sigma_z$ is negligible in comparison with in-plane stresses $\sigma_x$ and $\sigma_y$. Therefore, for a linearly elastic isotropic material, stresses $\sigma_x$ and $\sigma_y$ are related to strains $\varepsilon_x$ and $\varepsilon_y$ by the following constitutive relations:

$$\sigma_x = \frac{E}{(1-\mu^2)}(\varepsilon_x + \mu\varepsilon_y), \quad \sigma_y = \frac{E}{(1-\mu^2)}(\varepsilon_y + \mu\varepsilon_x). \tag{4}$$

   (b) The shear stresses $\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$ are related to shear strains $\gamma_{xy}$, $\gamma_{yz}$, $\gamma_{zx}$ by the following constitutive relations:

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx}. \tag{5}$$

4. The displacement $u$ in $x$-direction consists of bending component $u_b$ and shear component $u_s$. Similarly, the displacement $v$ in $y$-direction consists of bending component $v_b$ and shear component $v_s$:

$$u = u_b + u_s, \quad v = v_b + v_s. \tag{6}$$

(a) The bending component $u_b$ of displacement $u$ and $v_b$ of displacement $v$ are assumed to be analogous, respectively, to the displacements $u$ and $v$ given by CPT. Therefore, the expression for $u_b$ and $v_b$ can be given as

$$u_b = -z\frac{\partial w_b}{\partial x}, \tag{7}$$

$$v_b = -z\frac{\partial w_b}{\partial y}. \tag{8}$$

It may be noted that the displacement components $u_b$, $v_b$, and $w_b$ together do not contribute toward shear stresses $\tau_{zx}$ and $\tau_{yz}$.

(b) The shear component $u_s$ of displacement $u$ and the shear component $v_s$ of displacement $v$ are such that:
   (i) they give rise, in conjunction with $w_s$, to the parabolic variations of shear stresses $\tau_{zx}$ and $\tau_{yz}$ across the cross section of the plate in such a way that shear stresses $\tau_{zx}$ and $\tau_{yz}$ are zero at $z = -h/2$ and at $h/2$, and
   (ii) their contribution toward strains $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ is such that in the moments $M_x$, $M_y$, and $M_{xy}$ there is no contribution from the components $u_s$ and $v_s$.

### 3.2. Displacements, moments, shear forces in RPT

Based on the assumptions made in the preceding section, and going by the previous experience (Ref. [20]), it is possible to write shear component $u_s$ of displacements $u$, and shear component $v_s$ of displacements $v$ as:

$$u_s = h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial w_s}{\partial x}, \tag{9}$$

$$v_s = h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial w_s}{\partial y}. \tag{10}$$

Using expressions (3), and (6)–(10), one can write expressions for displacements $u$, $v$, $w$ as:

$$u(x,y,z,t) = -z\frac{\partial w_b}{\partial x} + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial w_s}{\partial x}, \tag{11}$$

$$v(x,y,z,t) = -z\frac{\partial w_b}{\partial y} + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial w_s}{\partial y}, \tag{12}$$

$$w(x,y,t) = w_b(x,y,t) + w_s(x,y,t). \tag{13}$$

Using expressions for displacements (11)–(13) in strain–displacement relations (2), the expressions for strains can be obtained:

$$\varepsilon_x = -z\frac{\partial^2 w_b}{\partial x^2} + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial^2 w_s}{\partial x^2}, \tag{14}$$

$$\varepsilon_y = -z\frac{\partial^2 w_b}{\partial y^2} + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial^2 w_s}{\partial y^2}, \tag{15}$$

$$\varepsilon_z = 0, \tag{16}$$

$$\gamma_{xy} = -2z\frac{\partial^2 w_b}{\partial x \partial y} + 2h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial^2 w_s}{\partial x \partial y}, \tag{17}$$

$$\gamma_{yz} = \left[\frac{5}{4} - 5\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial y}, \tag{18}$$

$$\gamma_{zx} = \left[\frac{5}{4} - 5\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x}. \tag{19}$$

Using expressions for strains from (14) to (19) in constitutive relations (4) and (5), the expressions for stresses can be obtained:

$$\sigma_x = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right) + \frac{Eh}{1-\mu^2}\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\left(\frac{\partial^2 w_s}{\partial x^2} + \mu\frac{\partial^2 w_s}{\partial y^2}\right), \tag{20}$$

$$\sigma_y = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right) + \frac{Eh}{1-\mu^2}\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\left(\frac{\partial^2 w_s}{\partial y^2} + \mu\frac{\partial^2 w_s}{\partial x^2}\right), \tag{21}$$

$$\tau_{xy} = -\frac{Ez}{1-\mu^2}(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y} + \frac{Eh}{1-\mu^2}(1-\mu)\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial^2 w_s}{\partial x \partial y}, \tag{22}$$

$$\tau_{yz} = \frac{E}{2(1+\mu)}\left[\frac{5}{4} - 5\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial y}, \tag{23}$$

$$\tau_{zx} = \frac{E}{2(1+\mu)}\left[\frac{5}{4} - 5\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x}. \tag{24}$$

The moments and shear forces are defined as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \int_{z=-h/2}^{z=h/2} \begin{Bmatrix} \sigma_x z \\ \sigma_y z \\ \tau_{xy} z \\ \tau_{zx} \\ \tau_{yz} \end{Bmatrix} \mathrm{d}z. \tag{25}$$

Using expressions for stresses (20)–(24) in (25), expressions for moments $M_x$, $M_y$ and $M_{xy}$ and shear forces $Q_x$ and $Q_y$ can be obtained. These expressions are:

$$M_x = -D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right], \tag{26}$$

$$M_y = -D\left[\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right], \tag{27}$$

$$M_{xy} = -D(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y}, \tag{28}$$

$$Q_x = \frac{5Eh}{12(1+\mu)}\frac{\partial w_s}{\partial x}, \tag{29}$$

$$Q_y = \frac{5Eh}{12(1+\mu)}\frac{\partial w_s}{\partial y}, \tag{30}$$

where the plate rigidity $D$ is defined by

$$D = \frac{Eh^3}{12(1-\mu^2)}. \tag{31}$$

It may be noted that expressions for moments $M_x$, $M_y$ and $M_{xy}$ contain only $w_b$ as an unknown function. Also, the expressions for shear forces $Q_x$ and $Q_y$ contain only $w_s$ as an unknown function.

### 3.3. Expressions for kinetic and strain energies

It should be noted that displacement $w$, given by Eq. (13), is not a function of $z$. As a result of this, normal strain $\varepsilon_z$ comes out to be zero. Therefore, for free vibration problem the expressions for kinetic energy $T$ and strain energy $U$ for three-dimensional body can be written as:

$$T = \int_{z=-h/2}^{z=h/2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \frac{1}{2}\rho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{32}$$

$$U = \int_{z=-h/2}^{z=h/2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \frac{1}{2}\left[\begin{matrix}\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy} \\ +\tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\end{matrix}\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{33}$$

Using expressions (11)–(24) in Eqs. (32) and (33), expressions for kinetic energy and strain energy can be written as:

$$\begin{aligned} T = {} & \frac{\rho h^3}{24}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left[\frac{\partial}{\partial t}\left(\frac{\partial w_b}{\partial x}\right)\right]^2 + \left[\frac{\partial}{\partial t}\left(\frac{\partial w_b}{\partial y}\right)\right]^2\right\}\mathrm{d}x\,\mathrm{d}y \\ & + \frac{\rho h^3}{2016}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left[\frac{\partial}{\partial t}\left(\frac{\partial w_s}{\partial x}\right)\right]^2 + \left[\frac{\partial}{\partial t}\left(\frac{\partial w_s}{\partial y}\right)\right]^2\right\}\mathrm{d}x\,\mathrm{d}y \\ & + \frac{\rho h}{2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\frac{\partial w_b}{\partial t} + \frac{\partial w_s}{\partial t}\right\}^2\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{34}$$

$$\begin{aligned} U = {} & \frac{Eh^3}{24(1-\mu^2)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\left(\frac{\partial^2 w_b}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_b}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_b}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 w_b}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \\ & + \frac{5Eh}{24(1+\mu)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\left(\frac{\partial w_s}{\partial x}\right)^2 + \left(\frac{\partial w_s}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \\ & + \frac{Eh^3}{2016(1-\mu^2)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\left(\frac{\partial^2 w_s}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_s}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_s}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 w_s}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{35}$$

### 3.4. Obtaining governing equations, boundary conditions in RPT by using Hamilton's principle

Governing differential equations and boundary conditions can be obtained using well-known Hamilton's principle

$$\int_{t_1}^{t_2} \delta(T-U)\,\mathrm{d}t = 0, \tag{36}$$

where $\delta$ indicates a variation w.r.t. $x$ and $y$ only; $t_1$, $t_2$ are values of time variable at the start and end of time interval (in the context of Hamilton's principle), respectively.

Using expressions (34) and (35) in the preceding equation and integrating the equation by parts, taking into account the independent variations of $w_b$ and $w_s$, yields the governing differential equations and boundary conditions.

### 3.4.1. Governing equations in RPT

The governing differential equations for free vibration of the plate are

$$D\nabla^2\nabla^2 w_b - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\left(\nabla^2 w_b\right) + \rho h\frac{\partial^2}{\partial t^2}(w_b + w_s) = 0, \tag{37}$$

$$\frac{1}{84}D\nabla^2\nabla^2 w_s - \frac{5Eh}{12(1+\mu)}\nabla^2 w_s - \frac{\rho h^3}{1008}\frac{\partial^2}{\partial t^2}(\nabla^2 w_s) + \rho h\frac{\partial^2}{\partial t^2}(w_b + w_s) = 0, \tag{38}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{39}$$

### 3.4.2. Boundary conditions in RPT

The boundary conditions of the plate are given as follows:

1. At corners $(x=0, y=0)$, $(x=0, y=b)$, $(x=a, y=0)$, and $(x=a, y=b)$ the following conditions hold:
   (a) The condition involving $w_b$ (i.e. bending component of lateral displacement)

$$-D\left[(1-\mu)\frac{\partial^2 w_b}{\partial x\partial y}\right] = 0 \quad \text{or } w_b \text{ is specified.} \tag{40}$$

   (b) The condition involving $w_s$ (i.e. shear component of lateral displacement)

$$-D\left[(1-\mu)\frac{\partial^2 w_s}{\partial x\partial y}\right] = 0 \quad \text{or } w_s \text{ is specified.} \tag{41}$$

2. On edges $x=0$ and $a$, the following conditions hold:
   (a) The conditions involving $w_b$ (i.e. bending component of lateral displacement)

$$-D\left[\frac{\partial^3 w_b}{\partial x^3} + (2-\mu)\frac{\partial^3 w_b}{\partial x\partial y^2}\right] + \frac{\rho h^3}{12}\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial x}\right)\right] = 0 \quad \text{or } w_b \text{ is specified,} \tag{42}$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right] = 0 \quad \text{or } \frac{\partial w_b}{\partial x} \text{ is specified.} \tag{43}$$

   (b) The conditions involving $w_s$ (i.e. shear component of lateral displacement)

$$\frac{5Eh}{12(1+\mu)}\frac{\partial w_s}{\partial x} - \frac{1}{84}D\left[\frac{\partial^3 w_s}{\partial x^3} + (2-\mu)\frac{\partial^3 w_s}{\partial x\partial y^2}\right] + \frac{\rho h^3}{1008}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_s}{\partial x}\right) = 0 \quad \text{or } w_s \text{ is specified,} \tag{44}$$

$$-D\left[\frac{\partial^2 w_s}{\partial x^2} + \mu\frac{\partial^2 w_s}{\partial y^2}\right] = 0 \quad \text{or } \frac{\partial w_s}{\partial x} \text{ is specified.} \tag{45}$$

3. On edges $y = 0$ and $b$, the following conditions hold:

(a) The conditions involving $w_b$ (i.e. bending component of lateral displacement)

$$-D\left[\frac{\partial^3 w_b}{\partial y^3}+(2-\mu)\frac{\partial^3 w_b}{\partial x^2\partial y}\right]+\frac{\rho h^3}{12}\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial y}\right)\right]=0 \quad \text{or } w_b \text{ is specified,} \tag{46}$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2}+\mu\frac{\partial^2 w_b}{\partial x^2}\right]=0 \quad \text{or} \quad \frac{\partial w_b}{\partial y} \text{ is specified.} \tag{47}$$

(b) The conditions involving $w_s$ (i.e. shear component of lateral displacement)

$$\frac{5Eh}{12(1+\mu)}\frac{\partial w_s}{\partial y}-\frac{1}{84}D\left[\frac{\partial^3 w_s}{\partial y^3}+(2-\mu)\frac{\partial^3 w_s}{\partial x^2\partial y}\right]+\frac{\rho h^3}{1008}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_s}{\partial y}\right)=0 \quad \text{or } w_s \text{ is specified,} \tag{48}$$

$$-D\left[\frac{\partial^2 w_s}{\partial y^2}+\mu\frac{\partial^2 w_s}{\partial x^2}\right]=0 \quad \text{or} \quad \frac{\partial w_s}{\partial y} \text{ is specified.} \tag{49}$$

## 4. Comments on RPT

1. With respect to governing equations, following can be noted:
   (a) In RPT, there are two governing equations [Eqs. (37) and (38)]. Both the governing equations are fourth-order partial differential equations.
   (b) The governing equations involve only two unknown functions (i.e. bending component $w_b$ and shear component $w_s$ of lateral deflection).
   Even theories of Reissner [1], Mindlin [3], which are first-order shear deformation theories and are considered to be simple ones, involve three unknown functions.
   It is interesting to note that Green also obtained governing equations for plate involving two unknown functions. The equations obtained by Green are quoted in Ref. [27, pp. 168–170]. But, Green's work was based on Reissner's approach and, therefore, the transverse shear stresses and shear strains do not exactly satisfy the constitutive relations. Whereas, in contrast, in RPT, these constitutive relations are exactly satisfied.
   (c) It should be noted here that certain assumptions of RPT (assumptions 4(bi) and 4(bii)) are quite different from those made in the work of Senthilnathan et al. [26]. Therefore, bending moments in RPT (expressions (26)–(28)) have no contribution from shearing component $w_s$, whereas it is not so in the theory proposed by Senthilnathan et al.
   It is emphasized here that governing equations of RPT are only inertial coupling, and there is no elastic coupling at all. In contrast, governing equations of the theory proposed by Senthilnathan et al. are elastically as well as inertially coupled.
   Additionally, an important point needs to be noted here is, shear forces [expressions (29) and (30)], of RPT can be written as

$$Q_x=\frac{5}{6}Gh\frac{\partial w_s}{\partial x}, \quad Q_y=\frac{5}{6}Gh\frac{\partial w_s}{\partial y}.$$

   It can be seen that the shear correction factor proposed by Reissner [2] is $\frac{5}{6}$ and by Mindlin [3] is $\pi^2/12$ (a value is very close to Reissner's $\frac{5}{6}$) is coming out automatically in the present theory. Whereas, if one works it out for the theory proposed by Senthilnathan et al., it comes out as $\frac{2}{3}$. Shear correction factor directly affects the shear contribution and thus accuracy.

2. With respect to boundary conditions, following can be noted:
   (a) There are two conditions per corner.
      i. One condition is stated in terms of $w_b$ and its derivatives only [i.e. condition (40)].
      ii. The remaining condition is stated in terms of $w_s$ and its derivatives only [i.e. condition (41)].
   (b) There are four boundary conditions per edge.
      i. Two conditions are stated in terms of $w_b$ and its derivatives only [e.g., in the case of edge $x = 0$, conditions (42) and (43)].
      ii. The remaining two conditions are stated in terms of $w_s$ and its derivatives only [e.g., in the case of edge $x = 0$, conditions (44) and (45)].
3. Some entities of RPT (e.g., a governing equation, moment expressions, boundary conditions) have strong similarity with those of CPT.
   (a) The following entities of RPT are identical, save for the appearance of the subscript, to the corresponding entities of the CPT:
      i. Moments $M_x$, $M_y$, $M_{xy}$ [i.e. expressions (26)–(28)].
      ii. Corner boundary condition [i.e. condition (40)].
      iii. Edge boundary conditions [i.e. conditions (42), (43), (46) and (47)].

   The bending component $w_b$ of lateral displacement figures in the just mentioned entities of RPT, whereas lateral displacement $w$ figures in the corresponding equations of the CPT.
   (b) One of the two governing Equations [i.e. Eq. (37)] has strong similarity with the governing equation of CPT. (If in Eq. (37) the term $\partial^2 w_s/\partial t^2$ is ignored, and if $w_b$ is replaced by $w$, then the resulting equation is identical to the governing equation of CPT).

It should be noted here that, as mentioned previously (assumption 4(a)), the bending component $w_b$ of transverse displacement, in case of RPT, is analogous to the transverse displacement given by CPT. Lee and Wang [24] discussed an important issue about the use of deflection components $w_b$ and $w_s$ in case of beam theory, one-dimensional counterpart of plate theory and deduced that the Timoshenko bending component $w_b$ is, in general, not the same as the Euler–Bernoulli deflection $w$.

In respect of boundary conditions, it is to be noted that $w_b$ together with $w_s$ must satisfy the physical displacement at the boundary.

## 5. RPT-Variant

It is possible to introduce simplification in RPT and yet retain very good accuracy. A simplified theory RPT-Variant can be obtained from RPT. This involves identifying and then ignoring terms of marginal utility from the expressions of strain energy and kinetic energy associated with RPT.

Assumptions of RPT-Variant are same as those of RPT. Expressions for displacements, strains, stresses, moments, shear forces in RPT-Variant are same as those of RPT. That is, expressions (11)–(30) are valid in RPT-Variant also.

The following needs to be noted:

1. The displacement $u$ has two components $u_b$ and $u_s$
   ($u$, $u_b$, $u_s$ are given by expressions (11), (7), and (9), respectively).
2. The displacement $v$ has two components $v_b$ and $v_s$
   ($v$, $v_b$, $v_s$ are given by expressions (12), (8), and (10), respectively).
3. (a) In the total kinetic energy, the energy component involving rotatory inertia constitutes only a small component.
   (b) From Eq. (32), expressions (11) and (12), it can be seen that in the kinetic energy, the rotatory inertia component arises because of the following terms:

$$\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2, \quad \frac{1}{2}\rho\left(\frac{\partial v}{\partial t}\right)^2.$$

As the displacement bending components $u_b$, $v_b$ are, in general, larger in magnitude than the corresponding displacement shear components $u_s$, $v_s$, respectively, the major contribution to rotatory inertia comes from terms arising out of displacement bending components $u_b$, $v_b$. Whereas, the contribution to rotatory inertia from terms arising out of displacement shear components $u_s$, $v_s$ is insignificant.
Therefore, in expression (34) for kinetic energy $T$, the contribution to rotatory inertia by the following term can safely be ignored:

$$\frac{\rho h^3}{2016}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left[\frac{\partial}{\partial t}\left(\frac{\partial w_s}{\partial x}\right)\right]^2+\left[\frac{\partial}{\partial t}\left(\frac{\partial w_s}{\partial y}\right)\right]^2\right\}\mathrm{d}x\,\mathrm{d}y.$$

4. (a) The displacement components $u_s$, $v_s$ do not contribute towards moments $M_x$, $M_y$, $M_{xy}$ (given by expressions (26)–(28), respectively).
As a result, the moment expressions do not contain any derivative of shear displacement component $w_s$ of lateral displacement.
Therefore, in the bending energy component of the strain energy, the contribution from terms involving $w_s$ is insignificant.
(b) From Eq. (33), it can be seen that in the strain energy $U$, there is bending energy component involving products of stresses $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ with corresponding strains $\varepsilon_x$, $\varepsilon_y$ and $\gamma_{xy}$. Such products contain entities which involve shear displacement component $w_s$ of lateral displacement. Any term containing such entities contributes insignificantly to the strain energy.
Therefore, in expression (35) for strain energy $U$, the contribution to the strain energy by the following term can safely be ignored:

$$\frac{Eh^3}{2016(1-\mu^2)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\left(\frac{\partial^2 w_s}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_s}{\partial y^2}\right)^2+2\mu\frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_s}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 w_s}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y.$$

Hence, expressions for kinetic energy $T$ and strain energy $U$ can be expressed with very good accuracy as follows:

$$T\approx\frac{\rho h^3}{24}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left\{\left[\frac{\partial}{\partial t}\left(\frac{\partial w_b}{\partial x}\right)\right]^2+\left[\frac{\partial}{\partial t}\left(\frac{\partial w_b}{\partial y}\right)\right]^2\right\}\mathrm{d}x\,\mathrm{d}y+\frac{\rho h}{2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\frac{\partial w_b}{\partial t}+\frac{\partial w_s}{\partial t}\right]^2\mathrm{d}x\,\mathrm{d}y, \tag{50}$$

$$\begin{aligned}U\approx{}&\frac{Eh^3}{24(1-\mu^2)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\left(\frac{\partial^2 w_b}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_b}{\partial y^2}\right)^2+2\mu\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_b}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 w_b}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\\&+\frac{5Eh}{24(1+\mu)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\left(\frac{\partial w_s}{\partial x}\right)^2+\left(\frac{\partial w_s}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{51}$$

### *5.1. Obtaining governing equations, boundary conditions in RPT-Variant by using Hamilton's principle*

Applying the Hamilton's principle as stated in Eq. (36), and using approximate expressions for kinetic energy $T$ and strain energy $U$ [i.e. expressions (50) and (51)], the governing differential equations and boundary conditions in respect of RPT-Variant can be obtained for free vibration of the plate.

*5.1.1. Governing equations in RPT-Variant*

$$D\nabla^2\nabla^2 w_b - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}(\nabla^2 w_b) + \rho h\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) = 0, \tag{52}$$

$$\frac{5Eh}{12(1+\mu)}\nabla^2 w_s - \rho h\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) = 0. \tag{53}$$

*5.1.2. Boundary conditions in RPT-Variant*

The boundary conditions of the plate are given as follows:

1. At corners $(x = 0, y = 0)$, $(x = 0, y = b)$, $(x = a, y = 0)$ and $(x = a, y = b)$ the following holds:

$$-D\left[(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y}\right] = 0 \quad \text{or } w_b \text{ is specified.} \tag{54}$$

2. On edges $x = 0$ and $a$, the following conditions hold:
   (a) The conditions involving $w_b$ (i.e. bending component of lateral displacement)

$$-D\left[\frac{\partial^3 w_b}{\partial x^3} + (2-\mu)\frac{\partial^3 w_b}{\partial x \partial y^2}\right] + \frac{\rho h^3}{12}\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial x}\right)\right] = 0 \quad \text{or } w_b \text{ is specified,} \tag{55}$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right] = 0 \quad \text{or} \quad \frac{\partial w_b}{\partial x} \text{ is specified.} \tag{56}$$

   (b) The condition involving $w_s$ (i.e. shear component of lateral displacement)

$$\frac{\partial w_s}{\partial x} = 0 \quad \text{or } w_s \text{ is specified.} \tag{57}$$

3. On edges $y = 0$ and $b$, the following conditions hold:
   (a) The conditions involving $w_b$ (i.e. bending component of lateral displacement)

$$-D\left[\frac{\partial^3 w_b}{\partial y^3} + (2-\mu)\frac{\partial^3 w_b}{\partial x^2 \partial y}\right] + \frac{\rho h^3}{12}\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial y}\right)\right] = 0 \quad \text{or } w_b \text{ is specified,} \tag{58}$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right] = 0 \quad \text{or} \quad \frac{\partial w_b}{\partial y} \text{ is specified.} \tag{59}$$

   (b) The condition involving $w_s$ (i.e. shear component of lateral displacement)

$$\frac{\partial w_s}{\partial y} = 0 \quad \text{or } w_s \text{ is specified.} \tag{60}$$

## 6. Comments on RPT-Variant

1. With respect to governing equations, following is to be noted:
   (a) In RPT-Variant, there are two governing equations.
      i. Eq. (52) is a fourth-order partial differential equation, whereas
      ii. Eq. (53) is a second-order partial differential equation.

(b) The governing equations involve only two unknown functions (i.e. bending component $w_b$ and shear component $w_s$ of lateral deflection).
Even theories of Reissner [2], Mindlin [3], which are first-order shear deformation theories and are considered to be simple ones, involve three unknown functions.
(c) RPT-Variant and, as has been noted earlier, RPT are the only theories, to the best knowledge of the authors, wherein in the governing equations, there is only inertial coupling, and there is no elastic coupling at all.

2. With respect to boundary conditions, following is to be noted:
(a) In RPT-Variant, there are three boundary conditions per edge.
Out of these, two conditions [e.g., in case of edge $x = 0$, Eqs. (55) and (56)] are stated in terms of $w_b$ and its derivatives only.
The remaining one condition [e.g., in case of edge $x = 0$, Eq. (57)] is stated in terms of $w_s$ and its derivative only.
(b) In RPT-Variant, there is one condition per corner, stated in terms of $w_b$ and its derivatives only [Eq. (54)].

3. Some entities of RPT-Variant have strong similarity with those of CPT:
(a) The following entities of RPT-Variant are identical, save for the appearance of the subscript, to the corresponding entities of the CPT:
i. Expressions for moments $M_x$, $M_y$, $M_{xy}$ [i.e. expressions (26)–(28)].
ii. Corner boundary condition [i.e. condition (54)].
iii. Edge boundary conditions involving bending component of lateral displacement [i.e. conditions (55), (56), (58) and (59)].

The bending component $w_b$ of lateral displacement figures in the just mentioned equations of RPT-Variant, whereas lateral displacement $w$ figures in the corresponding equations of the CPT.
(b) The governing Eq. (52), is very similar to the governing equation of CPT. [If in Eq. (52) the term $\partial^2 w_s/\partial t^2$ is ignored, and if $w_b$ is replaced by $w$, then the resulting equation is identical to the governing equation of CPT].

4. The governing equations of RPT-Variant are somewhat analogous to those obtained by Reissner's theory [2] and Mindlin's theory [3]. However, the following points are noteworthy:
(a) The governing equations of RPT-Variant involve only two unknown functions (i.e. $w_b$ and $w_s$), as against three in case of Reissner's theory [2] and Mindlin's theory [3]. (Though, this point has already been stated in preceding item 1(b), it is specifically mentioned here again so as to compare RPT-Variant with Reissner's theory and Mindlin's theory.)
(b) Because of strong similarity to the CPT, equations of RPT-Variant are easy to deal with.
(c) Moreover, in Mindlin's approach and Reissner's approach, the transverse shear stresses and shear strains do not exactly satisfy the constitutive relations. Whereas, in RPT-Variant, these constitutive relations are exactly satisfied.

## 7. Illustrative example: free vibrations of a simply-supported rectangular plate

An illustrative example would be taken up to demonstrate the effectiveness of RPT and RPT-Variant.

Consider a plate (of length $a$, width $b$, and thickness $h$) of homogeneous isotropic material. The plate occupies (in 0–$x$–$y$–$z$ right-handed Cartesian coordinate system) a region defined by Eq. (1). The plate has simply-supported boundary conditions at all four edges $x = 0$, $a$, and $y = 0$, $b$. Free vibrations of such a plate is also studied in other references (e.g., Refs. [12,16,28,29]).

### 7.1. Application of RPT for the illustrative example

RPT would now be applied to the illustrative example.

*7.1.1. Governing equations for the illustrative example when RPT is used*

The governing equations are the same as given by Eqs. (37) and (38).

*7.1.2. Boundary conditions for the illustrative example when RPT is used*

The boundary conditions of the plate are given as follows:

1. At corners $(x=0, y=0)$, $(x=0, y=b)$, $(x=a, y=0)$, and $(x=a, y=b)$ the following conditions hold:

$$w_b = 0, \tag{61}$$

$$w_s = 0. \tag{62}$$

2. On edges $x=0$ and $a$, the following conditions hold:

$$w_b = 0, \tag{63}$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right] = 0, \tag{64}$$

$$w_s = 0, \tag{65}$$

$$-D\left[\frac{\partial^2 w_s}{\partial x^2} + \mu\frac{\partial^2 w_s}{\partial y^2}\right] = 0. \tag{66}$$

3. On edges $y=0$ and $b$, the following conditions hold:

$$w_b = 0, \tag{67}$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right] = 0, \tag{68}$$

$$w_s = 0, \tag{69}$$

$$-D\left[\frac{\partial^2 w_s}{\partial y^2} + \mu\frac{\partial^2 w_s}{\partial x^2}\right] = 0. \tag{70}$$

*7.1.3. Solution of the illustrative example when RPT is used*

The following displacement functions $w_b$ and $w_s$ satisfy the boundary conditions (61)–(70):

$$w_b = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{b_{mn}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin(\omega_{mn}t), \tag{71}$$

$$w_s = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{s_{mn}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin(\omega_{mn}t), \tag{72}$$

where $W_{b_{mn}}$, $W_{s_{mn}}$ are constants and $\omega_{mn}$ is the circular frequency of vibration.

Using expressions (71) and (72) in the governing equations (37) and (38), one obtains following two equations for free vibration of plate:

$$\left\{\begin{array}{l} D\left[\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4\right] \\ \quad -\omega_{mn}^2\left[\frac{\rho h^3}{12}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}+\rho h\right]\end{array}\right\} W_{b_{mn}}-\{\omega_{mn}^2\rho h\}W_{s_{mn}}=0, \tag{73}$$

$$\{\omega_{mn}^2\rho h\}W_{b_{mn}}-\left\{\begin{array}{l}\frac{D}{84}\left[\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4\right] \\ \quad +\frac{5D(1-\mu)}{h^2}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right] \\ \quad -\omega_{mn}^2\left[\frac{\rho h^3}{1008}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}+\rho h\right]\end{array}\right\}W_{s_{mn}}=0. \tag{74}$$

A necessary and sufficient condition for the existence of a non-trivial solution for constants $W_{b_{mn}}$, $W_{s_{mn}}$ is that the determinant of the coefficient matrix is zero. This requirement gives following quadratic characteristic equation in $\omega_{mn}^2$ for RPT, which gives two eigenfrequencies for every combination of $m$ and $n$. Corresponding eigenvectors can be obtained. The characteristic equation is as follows:

$$\begin{aligned}&\left(\frac{\omega_{mn}^2\rho h^2}{G}\right)^2\left[(1-\mu)^2\left(\frac{1}{336}\alpha_{mn}+\frac{85}{28}\right)\right] \\ &-\left(\frac{\omega_{mn}^2\rho h^2}{G}\right)\left[(1-\mu)\left\{\begin{array}{l}\frac{1}{84}\alpha_{mn}^2 \\ \quad +\left(\frac{85}{14}+\frac{5(1-\mu)}{2}\right)\alpha_{mn} \\ \quad +30(1-\mu)\end{array}\right\}\right]+\left[\frac{1}{84}\alpha_{mn}^3+5(1-\mu)\alpha_{mn}^2\right] \\ &=0 \quad \text{for } m=1,2,\ldots,\infty \text{ and } n=1,2,\ldots,\infty,\end{aligned} \tag{75}$$

where

$$\alpha_{mn}=\left(\frac{m\pi h}{a}\right)^2+\left(\frac{n\pi h}{b}\right)^2.$$

From Eq. (75), one gets two frequencies for each combination of $m$ and $n$. Out of these two frequencies, the lower one is associated with predominantly bending mode, whereas the higher one is associated with predominantly shear mode.

The numerical results obtained, when $h/a=0.1$, for square plate ($b/a=1$), rectangular plate (having $b/a=\sqrt{2}$) are presented in Tables 1–3. The tables also present solutions obtained by other theories.

### 7.2. Application of RPT-Variant for the illustrative example

RPT-Variant would now be applied to the illustrative example.

#### 7.2.1. Governing equations for the illustrative example when RPT-Variant is used

The governing equations are the same as given by Eqs. (52) and (53).

*7.2.2. Boundary conditions for the illustrative example when RPT-Variant is used*

The boundary conditions of the plate are given as follows:

1. At corners $(x=0, y=0)$, $(x=0, y=b)$, $(x=a, y=0)$, and $(x=a, y=b)$ the following conditions hold:

$$w_b = 0. \tag{76}$$

2. On edges $x=0$ and $a$, the following conditions hold:

$$w_b = 0, \tag{77}$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu \frac{\partial^2 w_b}{\partial y^2}\right] = 0, \tag{78}$$

$$w_s = 0. \tag{79}$$

3. On edges $y=0$ and $b$, the following conditions hold:

$$w_b = 0, \tag{80}$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2} + \mu \frac{\partial^2 w_b}{\partial x^2}\right] = 0, \tag{81}$$

$$w_s = 0. \tag{82}$$

*7.2.3. Solution of the illustrative example when RPT-Variant is used*

The same displacement functions $w_b$ and $w_s$, used earlier and given by Eqs. (71) and (72), respectively, satisfy the boundary conditions (76)–(82). Using these displacement functions in governing equations (52) and (53), one gets the following two equations for free vibration of plate:

$$\left\{\begin{array}{l} D\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] \\ \quad -\omega_{mn}^2\left[\frac{\rho h^3}{12}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} + \rho h\right] \end{array}\right\} W_{b_{mn}} - \{\omega_{mn}^2 \rho h\} W_{s_{mn}} = 0, \tag{83}$$

$$\{\omega_{mn}^2 \rho h\} W_{b_{mn}} - \left\{\frac{5D(1-\mu)}{h^2}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] - \omega_{mn}^2 \rho h\right\} W_{s_{mn}} = 0. \tag{84}$$

Following the same procedure as that for RPT, we get the characteristic equation for RPT-Variant, which gives free vibration frequencies, as follows:

$$\left(\frac{\omega_{mn}^2 \rho h^2}{G}\right)^2 [3(1-\mu)^2] - \left(\frac{\omega_{mn}^2 \rho h^2}{G}\right)\left[\left(\frac{17-5\mu}{2}\right)\alpha_{mn}(1-\mu) + 30(1-\mu)^2\right] + [5(1-\mu)\alpha_{mn}^2] = 0$$
$$\text{for } m = 1, 2, \ldots, \infty \text{ and } n = 1, 2, \ldots, \infty, \tag{85}$$

where

$$\alpha_{mn} = \left(\frac{m\pi h}{a}\right)^2 + \left(\frac{n\pi h}{b}\right)^2.$$

From Eq. (85), one gets two frequencies for each combination of $m$ and $n$. Out of these two frequencies, the lower one is associated with predominantly bending mode, whereas the higher one is associated with predominantly shear mode.

The numerical results obtained, when $h/a = 0.1$, for square plate ($b/a = 1$), rectangular plate (having $b/a = \sqrt{2}$) are presented in Tables 1–3. The tables also present solutions obtained by other theories.

## 8. Discussion on results

Results using RPT and RPT-Variant are tabulated in Tables 1–3. The tables also present solutions obtained using exact theory [16], Reddy's theory (termed as 'higher-order shear deformation plate theory' (HSDPT) in Refs. [12,28]), Mindlin's theory (termed as 'first-order shear deformation plate theory' (FSDPT) in Refs. [12,28]) and CPT taking into account rotary inertia (as reported in Refs. [12,28]).

It needs to be mentioned here that the non-dimensional frequency $\overline{\omega}$ as defined in Ref. [28] has a misprint in it. Reddy, in Ref. [12], has given the proper definition of non-dimensional frequency $\{\overline{\omega} = \omega h(\sqrt{\rho/G})\}$. Also, the non-dimensional frequencies for rectangular plate (having $b/a = \sqrt{2}$) as reported by Reddy [12], Reddy and Phan [28] and by Lee and Reismann [31], have typographical errors in the placement of decimal symbol and this can be verified by using results later given by Wang et al. [30]. However, in interpreting the results given in Ref. [30], it needs to be noted that, in that reference, the definition of non-dimensional frequency involves plate rigidity $D$ (instead of use of shear modulus of elasticity $G$ in Refs. [12,16,28,31]).

In Tables 1–3, the % error quoted against a particular theory is calculated with reference to the exact theory of Ref. [16].

Also, results quoted are in ascending order of frequencies. The frequency indicated, when $m = 1$ and $n = 1$, is the fundamental frequency.

In Table 1, results for predominantly bending mode frequencies for a square plate ($b/a = 1$, $h/a = 0.1$) are given.

In Table 2, results for predominantly bending mode frequencies for a rectangular plate ($b/a = \sqrt{2}$, $h/a = 0.1$) are given.

In Table 3, results for predominantly shear mode frequencies for a square plate ($b/a = 1$, $h/a = 0.1$) are given.

The following observations can be made:

1. In respect of predominantly bending mode frequencies, the following can be said from Tables 1 and 2:
   (a) RPT, for both square and rectangular plates, gives very good accuracy (e.g., for the square plate, for a mode when $m = 4$ and $n = 4$, the error is $-0.96\%$; and for rectangular plate, for a mode when $m = 2$ and $n = 5$, the error is $-0.68\%$).
   Whereas, Reddy's theory gives marginally accurate results than RPT (e.g., for the square plate, for a mode when $m = 4$ and $n = 4$, the error is $-0.39\%$; and for rectangular plate, for a mode when $m = 2$ and $n = 5$, the error is $-0.38\%$).
   It should be noted that RPT involves only two unknown functions and two differential equations as against five unknown functions and five differential equations in case of Reddy's theory.
   Moreover, in RPT, both the differential equations are only inertially coupled and there is no elastic coupling; and, therefore, the equations are easier to solve.
   Whereas, in Reddy's theory, all the five differential equations are inertially as well as elastically coupled, and therefore, these equations are difficult to solve.
   CPT results are not satisfactory (e.g., for the square plate, for a mode when $m = 4$ and $n = 4$, the error is 25.96%; and for rectangular plate, for a mode when $m = 2$ and $n = 5$, the error is 16.43%).
   CPT involves use of only one unknown function and only one differential equation. Compared to CPT, use of RPT involves only one additional function and one additional number of differential equation. But, when RPT is used, gain in accuracy of results is substantial.
   (b) RPT-Variant, for both square and rectangular plates, gives good accuracy (e.g., for the square plate, for a mode when $m = 4$ and $n = 4$, the error is $-1.15\%$; and for rectangular plate, for a mode when $m = 2$ and $n = 5$, the error is $-0.75\%$).
   Incidentally, though Mindlin's theory also gives more or less same results as given by RPT-Variant, it should be noted that
      i. RPT-Variant involves only two unknown functions and two differential equations as against three unknown functions and three differential equations in case of Mindlin's theory.

Table 1
Comparison of non-dimensional natural predominantly bending mode frequencies $\overline{\omega}_{mn}$ of simply-supported isotropic square plate $\overline{\omega}_{mn} = \omega_{mn} h\sqrt{\rho/G}$; $h/a = 0.1$, $b/a = 1.0$

| m | n | Non-dimensional natural frequency $\overline{\omega}_{mn}$ and corresponding % error (quoted in brackets) | | | | | |
|---|---|---|---|---|---|---|---|
| | | EXACT [28] | Reddy [28] | Mindlin [28] | CPT [28] | RPT | RPT-Variant |
| 1 | 1 | 0.0932 | 0.0931 | 0.0930 | 0.0955 | 0.0930 | 0.0930 |
| | | (0.00) | (−0.11) | (−0.22) | (2.47) | (−0.22) | (−0.22) |
| 1 | 2 | 0.2226[a] | 0.2222 | 0.2219 | 0.2360 | 0.2220 | 0.2219 |
| | | (0.00) | (−0.18) | (−0.32) | (6.02) | (−0.27) | (−0.32) |
| 2 | 2 | 0.3421 | 0.3411 | 0.3406 | 0.3732 | 0.3406 | 0.3406 |
| | | (0.00) | (−0.29) | (−0.44) | (9.09) | (−0.44) | (−0.44) |
| 1 | 3 | 0.4171 | 0.4158 | 0.4149 | 0.4629 | 0.4151 | 0.4149 |
| | | (0.00) | (−0.31) | (−0.53) | (10.98) | (−0.48) | (−0.53) |
| 2 | 3 | 0.5239 | 0.5221 | 0.5206 | 0.5951 | 0.5208 | 0.5206 |
| | | (0.00) | (−0.34) | (−0.63) | (13.02) | (−0.59) | (−0.63) |
| 1 | 4 | — | 0.6545 | 0.6520 | 0.7668 | 0.6525 | 0.6520 |
| | | — | — | — | — | — | — |
| 3 | 3 | 0.6889 | 0.6862 | 0.6834 | 0.8090 | 0.6840 | 0.6834 |
| | | (0.00) | (−0.39) | (−0.80) | (17.43) | (−0.71) | (−0.80) |
| 2 | 4 | 0.7511 | 0.7481 | 0.7446 | 0.8926 | 0.7454 | 0.7447 |
| | | (0.00) | (−0.40) | (−0.87) | (18.84) | (−0.76) | (−0.85) |
| 3 | 4 | — | 0.8949 | 0.8896 | 1.0965[a] | 0.8908 | 0.8897 |
| | | — | — | — | — | — | — |
| 1 | 5 | 0.9268 | 0.9230 | 0.9174 | 1.1365 | 0.9187 | 0.9174 |
| | | (0.00) | (−0.41) | (−1.01) | (22.63) | (−0.87) | (−1.01) |
| 2 | 5 | — | 1.0053[a] | 0.9984 | 1.2549 | 1.0001 | 0.9984 |
| | | — | — | — | — | — | — |
| 4 | 4 | 1.0889 | 1.0847 | 1.0764 | 1.3716 | 1.0785 | 1.0764 |
| | | (0.00) | (−0.39) | (−1.15) | (25.96) | (−0.96) | (−1.15) |
| 3 | 5 | — | 1.1361 | 1.1268 | 1.4475 | 1.1292 | 1.1269 |
| | | — | — | — | — | — | — |

$$\% \text{ error} = \left(\frac{\text{value obtained by a theory}}{\text{corresponding value by exact theory}} - 1\right) \times 100.$$

— against an entry indicates that results/data are not available.

[a]Converted in the present non-dimensional form from results quoted in Ref. [30] (as the corresponding results in Ref. [28] have some misprints).

ii. unlike Mindlin's theory, RPT-Variant satisfies:
   A. shear stress free boundary conditions,
   B. constitutive relations in respect of transverse shear stresses and strains (and, therefore, does not require shear correction factor).

(c) The following interesting observation can be made from Tables 1 and 2: Frequencies for predominantly bending modes obtained by all the theories, mentioned in the tables, with the exception of CPT, are lower than the corresponding frequencies obtained by exact theory.
The present authors are unable to offer reasons for this observation. Apparently, to the best knowledge of the authors, there are no discussions in the literature about such observations in respect of Mindlin's theory and Reddy's theory.

2. RPT has two variables in displacement function so we get two frequencies for every combination of $m$ and $n$. Lower one is predominantly bending mode and other is predominantly shear mode. These predominantly shear mode frequencies are very high, and, are important for shear wave studies.
Table 3 presents predominantly shear mode frequencies for square plate, obtained by RPT, RPT-Variant, Mindlin's theory (as reported in Ref. [16]), exact results by Srinivas [16]. Reddy's theory can give five frequencies for every combination of $m$ and $n$, but in Refs. [12,28] predominantly shear mode frequencies are not given.

Table 2
Comparison of non-dimensional natural predominantly bending mode frequencies $\overline{\omega}_{mn}$ of simply-supported isotropic rectangular plate $\overline{\omega}_{mn} = \omega_{mn} h\sqrt{\rho/G}$; $h/a = 0.1$, $b/a = \sqrt{2}$

| $m$ | $n$ | Non-dimensional natural frequency $\overline{\omega}_{mn}$ and corresponding % error (quoted in brackets) | | | | | |
|---|---|---|---|---|---|---|---|
| | | EXACT [30][a] | Reddy [30][a] | Mindlin [30][a] | CPT [30][a] | RPT | RPT-Variant |
| 1 | 1 | 0.0704 | 0.07038 | 0.07036 | 0.07180 | 0.07036 | 0.07036 |
| | | (0.00) | (−0.03) | (−0.06) | (1.99) | (−0.06) | (−0.06) |
| 1 | 2 | 0.1376 | 0.13738 | 0.13729 | 0.14273 | 0.1373 | 0.13729 |
| | | (0.00) | (−0.16) | (−0.23) | (3.73) | (−0.22) | (−0.23) |
| 2 | 1 | 0.2018 | 0.20141 | 0.20123 | 0.21281 | 0.20124 | 0.20123 |
| | | (0.00) | (−0.19) | (−0.28) | (5.46) | (−0.28) | (−0.28) |
| 1 | 3 | 0.2431 | 0.24263 | 0.24235 | 0.25908 | 0.24238 | 0.24236 |
| | | (0.00) | (−0.19) | (−0.31) | (6.57) | (−0.30) | (−0.30) |
| 2 | 2 | 0.2634 | 0.26283 | 0.26250 | 0.28207 | 0.26254 | 0.26251 |
| | | (0.00) | (−0.22) | (−0.34) | (7.09) | (−0.33) | (−0.34) |
| 2 | 3 | 0.3612 | 0.36013 | 0.35948 | 0.39575 | 0.35957 | 0.35948 |
| | | (0.00) | (−0.30) | (−0.48) | (9.57) | (−0.45) | (−0.48) |
| 1 | 4 | 0.3800 | 0.37891 | 0.37818 | 0.41822 | 0.37828 | 0.37818 |
| | | (0.00) | (−0.29) | (−0.48) | (10.05) | (−0.45) | (−0.48) |
| 3 | 1 | 0.3987 | 0.39748 | 0.39666 | 0.44062 | 0.39678 | 0.39667 |
| | | (0.00) | (−0.31) | (−0.51) | (10.51) | (−0.48) | (−0.51) |
| 3 | 2 | 0.4535 | 0.45198 | 0.45089 | 0.50729 | 0.45106 | 0.45090 |
| | | (0.00) | (−0.34) | (−0.58) | (11.86) | (−0.54) | (−0.57) |
| 2 | 4 | 0.4890 | 0.48737 | 0.48608 | 0.55133 | 0.48629 | 0.48609 |
| | | (0.00) | (−0.33) | (−0.60) | (12.75) | (−0.55) | (−0.60) |
| 3 | 3 | 0.5411 | 0.53915 | 0.53754 | 0.61680 | 0.53782 | 0.53754 |
| | | (0.00) | (−0.36) | (−0.66) | (13.99) | (−0.61) | (−0.66) |
| 1 | 5 | 0.5411 | 0.53915 | 0.53754 | 0.61680 | 0.53782 | 0.53754 |
| | | (0.00) | (−0.36) | (−0.66) | (13.99) | (−0.61) | (−0.66) |
| 2 | 5 | 0.6409 | 0.63846 | 0.63609 | 0.74563 | 0.63654 | 0.63610 |
| | | (0.00) | (−0.38) | (−0.75) | (16.43) | (−0.68) | (−0.75) |

$$\%\ \text{error} = \left(\frac{\text{value obtained by a theory}}{\text{corresponding value by exact theory}} - 1\right) \times 100.$$

[a]Converted in the present non-dimensional form from results quoted in Ref. [30].

For predominantly shear mode frequencies also, both RPT and RPT-Variant give good accuracy in results (e.g., for the square plate, for a mode when $m = 4$ and $n = 4$, in case of RPT the error is 2.45%, and in case of RPT-Variant the error is 2.86%). In case of Mindlin's theory, the corresponding error is 2.47%. Table 3 shows that Mindlin's theory gives marginally more accuracy than RPT for some lower shear modes up-to the mode wherein $m = 2$ and $n = 3$. But for higher shear modes RPT gives marginally improved results over Mindlin's theory. The reason can be attributed to the particular value of shear correction factor used in Mindlin's theory. It should be noted that both RPT as well as RPT-Variant do not require shear correction factor.

## 9. Concluding remarks

In this paper, two variable refined plate theory (RPT) has been applied for free vibrations analysis of isotropic plate. Its simplified version (RPT-Variant) is also presented.

1. The following points need to be noted in respect of RPT:
   (a) The theory is variationally consistent and use of the theory results in two fourth-order governing differential equations, which are only inertially coupled and there is no elastic coupling at all.

Table 3
Comparison of non-dimensional natural predominantly shear mode frequencies $\overline{\omega}_{mn}$ of simply-supported isotropic square plate $\overline{\omega}_{mn} = \omega_{mn} h\sqrt{\rho/G}$; $h/a = 0.1$, $b/a = 1.0$

| $m$ | $n$ | Non-dimensional natural frequency $\overline{\omega}_{mn}$ and corresponding % error (quoted in brackets) | | | |
|---|---|---|---|---|---|
| | | EXACT [16] | Mindlin [16] | RPT | RPT-Variant |
| 1 | 1 | 3.2465<br>(0.00) | 3.25380<br>(0.23) | 3.25552<br>(0.28) | 3.27411<br>(0.85) |
| 1 | 2 | 3.3933<br>(0.00) | 3.41120<br>(0.53) | 3.41250<br>(0.57) | 3.43102<br>(1.11) |
| 2 | 2 | 3.5298<br>(0.00) | 3.5580<br>(0.80) | 3.55894<br>(0.83) | 3.57742<br>(1.35) |
| 1 | 3 | 3.6160<br>(0.00) | 3.65100<br>(0.97) | 3.65173<br>(0.99) | 3.67017<br>(1.50) |
| 2 | 3 | 3.7393<br>(0.00) | 3.78420<br>(1.20) | 3.78473<br>(1.22) | 3.80317<br>(1.71) |
| 1 | 4 | —<br>— | —<br>— | 3.95421<br>— | 3.97075<br>— |
| 3 | 3 | 3.9310<br>(0.00) | 3.99260<br>(1.57) | 3.99280<br>(1.57) | 4.01113<br>(2.04) |
| 2 | 4 | 4.0037<br>(0.00) | 4.07200<br>(1.71) | 4.07195<br>(1.70) | 4.09026<br>(2.16) |
| 3 | 4 | —<br>— | —<br>— | 4.26131<br>— | 4.27953<br>— |
| 1 | 5 | 4.2099<br>(0.00) | 4.29820<br>(2.10) | 4.29786<br>(2.09) | 4.31607<br>(2.52) |
| 2 | 5 | —<br>— | —<br>— | 4.40518<br>— | 4.42334<br>— |
| 4 | 4 | 4.4013<br>(0.00) | 4.50980<br>(2.47) | 4.50923<br>(2.45) | 4.52735<br>(2.86) |
| 3 | 5 | —<br>— | —<br>— | 4.57694<br>— | 4.59502<br>— |

$$\%\ \text{error} = \left(\frac{\text{value obtained by a theory}}{\text{corresponding value by exact theory}} - 1\right) \times 100.$$

— against an entry indicates that results/data are not available.

RPT-Variant also shares these features. No other theory, to the best of the knowledge of the authors, has these feature.

(b) Number of unknown functions involved in the theory is only two. Even in the Mindlin's theory (a first-order shear deformation theory), three unknown functions are involved.
(c) The theory has strong similarity with the classical plate theory in many aspects (in respect of a governing equation, boundary conditions, moment expressions).
(d) i. The theory assumes displacements such that transverse shear stress variation is realistic (giving shear stress free surfaces and parabolic variation of shear stress across the thickness).
ii. Constitutive relations in respect of shear stresses and shear strains are satisfied (and, therefore, shear correction factor is not required).
(e) The CPT comes out as a special case of formulations. Therefore, the finite elements based on the theory will be free from shear locking.
(f) The results obtained using RPT is found to be in excellent agreement with the exact theory. The gain in accuracy obtained by using Reddy's theory (which is more involved than the RPT) is only marginal.

2. Whatever has been just mentioned about RPT, is true for RPT-Variant as well, except for the following specifics:
(a) It is derived from RPT by not taking into account entities of marginal utility. As a result, use of the theory results in having two differential equations, wherein one is a fourth-order differential equation, and another one is a second-order differential equation.

(b) The results obtained using RPT-Variant is found to be in very good agreement with the exact theory. The gain in accuracy obtained by using Mindlin's theory (which is more involved than the RPT) is only marginal.

In conclusion, it can be said that for vibration problems, RPT and RPT-Variant can be successfully utilized for simplicity as well as accuracy.

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